Indian Statistical Institute
Bangalore Centre
B.Math (Hons.) Second Year 2009-2010
Second Semester
Semester Examination
Statistics II

26.04.10 Time :- 3 hours

Answer as much as you can. The maximum you can score is 120
The notation used have their usual meaning unless stated otherwise.
State clearly the results that you assume.

1. (a) Consider a family of distributions $f(x, \theta), \theta \in \Theta$. Define (i) sufficient statistics, (ii) complete statistic and (iii) ancillary statistic for θ.

   (b) Suppose $X_1, X_2, \cdots X_n$ is a random sample from $f(x, \theta) = (1/\theta)exp(-x/\theta)$. Show that

   (i) $T(X) = \sum_{i=1}^{n} X_i$ is sufficient for θ.

   (ii) $U(X) = X_n/T(X)$ is an ancillary statistics for θ.

   (c) If T is a complete sufficient statistics for a family of distributions $f(x, \theta), \theta \in \Theta$ and V is an ancillary statistic then show that V is independent of T.

   (d) Assuming that $T(X)$ of Q(b)(i) is complete, find $E[U(X)]$.

   [3x3 + (3+5) + 7 + 10 = 34]

2. Suppose $X_1, X_2, \cdots X_n$ is a random sample from a normal (μ, σ^2) population. Let $\underset{\sim}{X} = (X_1, X_2, \cdots X_n)$.

   (a) Find maximum likelihood estimators of μ and σ^2 and check whether they are unbiased.

   (b) Prove that $T(X) = \bar{X}$ and $U(X) = \sum_{i=1}^{n}(X_i - \bar{X})^2$ are independent. [Hint : use an orthogonal transformation $\underset{\sim}{Y} = O\underset{\sim}{X}$, such that $Y_1 = \sqrt{n}\bar{X}$.]

   (c) Find the distribution of $U(X) = \sum_{i=1}^{n}(X_i - \bar{X})^2/\sigma^2$. Using this or otherwise find an unbiased estimator of $(\sigma^2)^{-1}$.

   (d) Find an unbiased estimator of μ^2.

   (e) Find an unbiased estimator of μ^2/σ^2. [Hint : Use (b) and (c)].

   [(4+4) + 6 + (5+5) + 6 + 8 = 38]

3. (a) Define (a) the level and the (b) the size of a test. When is a test said to be more powerful than another one?

   (b) Consider a random variable X with p.d.f $f(x, \theta)$. Consider the testing problem $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$.

   (i) Show that there always exist a size α test ϕ which rejects H_0 if the ratio $f(x, \theta_1)/f(x, \theta_0)$ is too large.

   (ii) Show that ϕ of Q(i) is most powerful among all size α tests.

   [3x2 + (6+10) = 22]

4. (a) When is a family of densities $f(x, \theta),\ \theta \in \Theta$ said to have monotone likelihood ratio (MLR) in $T(X)$? When is a test said to be uniformly most powerful?

   (b) Suppose the family of densities $f(x, \theta),\ \theta \in \Theta$ have MLR in $T(X)$? Consider the testing problem

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta > \theta_0.$$

Derive an uniformly most powerful test of size α test for this problem.

[4x2 + 12 =20]

5. Define uniformly most accurate (UMA) lower confidence bound with confidence level $1 - \alpha$ of a parameter θ. Suppose X is a continuous random variable having p.d.f $f(x, \theta)$. Suppose we want to find an UMA lower confidence bound for θ with confidence level 0.95. Consider an appropriate one-sided testing problem for θ. Show how you can find the required UMA lower confidence bound using the uniformly most powerful test of level 0.05 for this testing problem.

[4 + 14=18]